# Amalgamated Worksheet # 1

Various Artists

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## 1 Mike Hartglass

*For all exercises, $V$ is a finite dimensional complex vector space over $\mathbb{C}$*
1.) Prove that if $T \in \mathcal{L}(V)$ has only one eigenvalue, then every vector $v \in V$ is a generalized eigenvector of $T$ (*Hint*: Use the Jordan decomposition of $T$).

2.) For this problem, suppose that $S$ and $T$ are operators on a finite dimensional complex vector space $V$.

$a$.) Suppose that $ST$ is nilpotent. Prove that $TS$ is nilpotent.

$b$.) Suppose $S$ and $T$ are nilpotent and $ST = TS$. Prove that $S + T$ is nilpotent.

$c$.) Suppose $S$ and $T$ are nilpotent. Must $S + T$ be nilpotent? Give a proof or give a counterexample.

3.) Let $V$ be an $n-$dimensional complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, ..., \lambda_m$ be the (distinct) eigenvalues of $T$ (hence $m \leq n$). We know from class that if $U_k = \text{Null}(T - \lambda_k I)^n$, we have

$$V = U_1 \oplus \cdots \oplus U_n.$$

$a$.) Prove that each $U_k$ is invariant under $T$.

$b$.) Prove that $T - \lambda_k I$ restricted to $U_k$ is nilpotent.

$c$.) Consider $E_i \in \mathcal{L}(V)$ defined by $E_i(v_1 + v_2 + \cdots + v_m) = v_i$ whenever $v_k \in U_k$ (notice that this is well defined by the direct sum decomposition). Prove that $T$ commutes with each $E_i$.

$d$.) Use the $E_i's$ to show that we can write $T = D + N$ where $D$ is diagonalizable and $N$ is nilpotent with $DN = ND$.

# 2 Peyam Tabrizian

## Problem 1:

Find all the generalized eigenvectors of $T \in \mathcal{L}(\mathbb{R}^3)$ defined by:

$$T(x, y, z) = (x + y + z, y + z, z)$$

## Problem 2:

Suppose that $T \in \mathcal{L}(V)$ has $n$ distinct eigenvalues (where $n = dim(V)$), and that $S \in \mathcal{L}(V)$ has the same eigenvectors as $T$ (but not necessarily with the same eigenvalues). Show that $ST = TS$.

## Problem 3:

Show that if $V$ is a vector space over $\mathbb{C}$ and if 0 is the only eigenvalue of $T \in \mathcal{L}(V)$, then $T$ is nilpotent

## Problem 4:

Show that if $Nul(T - \lambda I) = Nul((T - \lambda I)^2)$, then $V$ has a basis of eigenvectors of $T$ (that is, $T$, is diagonalizable)

## Problem 5:

(if time permits) Suppose $T \in \mathcal{L}(V)$

(a) Show that $T(T - \lambda I)^n = (T - \lambda I)^n T$.

(b) Use $(a)$ to show that $(T - \lambda I)(T - \mu I)^n = (T - \mu I)^n (T - \lambda I)$.

**Hint:** For $(a)$, expand $(T - \lambda I)^n$ out, using the fact that for some scalars $a_i$, we have:

$$(A + B)^k = \sum_{i=0}^{k} a_i \, A^i B^{k-i}$$

(this is called the binomial formula. Technically $a_i = \frac{k!i!}{(k-i)!}$, but you won't need this)

**Note:** More generally, using induction, one can show (but you don't have to) that:

$$T^m (T - \lambda I)^n = (T - \lambda I)^n T^m$$

and that

$$(T - \lambda I)^m (T - \mu I)^n = (T - \mu I)^n (T - \lambda I)^m$$

where $m = 0, 1, \cdots$. Those facts are used in part 3 of Axler's paper.